

A THEOREM ON QUADRATIC FORMS AND ITS APPLICATION IN THE CALCULUS OF VARIATIONS*

BY

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1. **Principal result.** Let $P(z)$, $Q_1(z)$, $\dots$, $Q_r(z)$ be quadratic forms in the real variables $z_1, \dots, z_m$ with real coefficients having the following properties: the form $P(z)$ is positive at each point $(z) \neq (0)$ at which the forms $Q_1(z), \dots, Q_r(z)$ vanish simultaneously; for every set of constants $S_1, \dots, S_r$, not all zero, the quadratic form $S_i Q_i(z)^\dagger$ is indefinite; for every linear subspace L on which the quadratic forms $Q_1, \dots, Q_r$ do not vanish simultaneously, save at $(z) = (0)$, there is a linear combination $S_i Q_i(z)$ of these forms which is positive definite on L . The principal result given in the present note is given in the following:

THEOREM 1. *Under the above hypotheses there exists a set of constants $S_1, \dots, S_r$ such that the quadratic form*

$$P(z, S) = P(z) + S_i Q_i(z) \quad (i = 1, \dots, r)$$

is positive definite in the variables $z_1, \dots, z_m$. The last two hypotheses may be dropped when $r = 1$.

This theorem was proposed in a somewhat different form by Bliss in 1937 in a seminar on the calculus of variations. It is useful in sufficiency proofs for multiple integral problems. Proofs of the theorem for the case $r = 1$ were given in the seminar by Albert,[‡] Reid,[§] McShane and Hestenes, each using a different method. The last two of these proofs have not been published. The proof of Theorem 1 here given is due to McShane and is an immediate extension of the one given by McShane for the case $r = 1$. In the next section Theorem 1 will be applied to the case in which $(z) = (x_1, \dots, x_n, y_1, \dots, y_n)$ and the forms $Q_1, \dots, Q_r$ are the two-rowed minors of the $n \times 2$ -dimensional matrix $(x_k y_k)$. If $n = 2$ then $r = 1$ and Theorem 2 below is an immediate consequence of Theorem 1. If $n > 2$ the further result described in Theorem 3 below is

* Presented separately to the Society, by McShane on April 8, 1939, and by M. R. Hestenes on December 29, 1939; received by the editors November 29, 1939.

† A repeated index denotes summation.

‡ A. A. Albert, *A quadratic form problem in the calculus of variations*, Bulletin of the American Mathematical Society, vol. 44 (1938), pp. 250–252.

§ W. T. Reid, *A theorem on quadratic forms*, Bulletin of the American Mathematical Society, vol. 44 (1938), pp. 437–440.

needed. This further result for the case $n=3$ has been established by McShane.* The proofs of Theorems 3, 4 and 5 here given are due to Hestenes.†

The proof of Theorem 1 is based on two lemmas, the first of which is the following:

LEMMA A. *Let $m(S)$ be the minimum of the quadratic form $P(z, S)$ on the unit $(m-1)$ -sphere $z_k z_k = 1$. There exists a set of constants (S_0) which maximizes the function $m(S)$.*

For a set of constants $S_1, \dots, S_r$ let $M(S)$ be the maximum of the quadratic form $-S_i Q_i$ on the unit sphere $z_k z_k = 1$. We have $M(S) > 0$ if $(S) \neq (0)$ since the quadratic form $-S_i Q_i$ is indefinite by hypothesis. Moreover $M(hS) = hM(S)$ for every positive number h . Let M_0 be the minimum of $M(S)$ on the set $S_i S_i = 1$. Since this minimum is attained, we have $M_0 > 0$. Finally let N be the maximum of $P(z)$ on the set $z_k z_k = 1$. Then for every set of constants (S) there is a point (z) such that $z_k z_k = 1$, $-S_i Q_i(z) = M(S) \geq hM_0$, $S_i S_i = h^2$ and

$$m(S) \leq P(z, S) \leq N - M(S) \leq N - hM_0.$$

It follows that there is a positive constant h_0 such that when $h > h_0$ we have $m(S) < m(0)$ for every set (S) with $S_i S_i = h^2$. There is accordingly a set (S_0) such that the relation $m(S) \leq m(S_0)$ holds for every set (S) , as was to be proved.

LEMMA B. *Let (S_0) be a set of constants which maximizes the function $m(S)$ and set $m_0 = m(S_0)$. The set of points (z) satisfying the equation $P(z, S_0) = m_0 z_k z_k$ forms a linear space L . There is no quadratic form $S_i Q_i$ which is positive definite on L .*

The set L consists of all points z at which the function $P(z, S_0) - m_0 z_k z_k$ attains its minimum value 0. Hence the partial derivatives of that function vanish on L , and therefore the equations

$$(1) \quad P_{z_k}(z, S_0) = 2m_0 z_k$$

hold for all z in L . Conversely, if we multiply equations (1) by z_k and sum on k we see that every point z which satisfies (1) lies in L . Thus L is the set of solutions of the linear equations (1), and is therefore linear. Suppose now that there exists a quadratic form $Q = S_i Q_i$ that is positive definite on L . Let K be the unit sphere $z_k z_k = 1$ and L_1 the set of points in L on K . Choose $b > 0$ such

* E. J. McShane, *The condition of Legendre for double integral problems of the calculus of variations*, Bulletin of the American Mathematical Society, abstract 45-5-209.

† M. R. Hestenes, *A theorem on quadratic forms and its application in the calculus of variations*, Bulletin of the American Mathematical Society, abstract 46-1-83.

that $Q(z) > b$ on L_1 , and let N be a neighborhood of L_1 relative to K on which $Q(z) > b$. Let m_1 be the minimum of $P(z, S_0)$ on the closed set $K - N$. Then $m_1 > m_0$. It follows that for a sufficiently small positive constant h one will have

$$P(z, S_0 + hS) = P(z, S_0) + hQ > m_0$$

on $K - N$. But $P(z, S_0 + hS) > m_0 + hb$ on N , and hence $m(S_0 + hS) > m(S_0)$, contrary to our choice of the set (S_0) . This proves Lemma B.

In order to prove Theorem 1 we note that there is a point $(z) \neq (0)$ on the set L described above at which the forms $Q_1, \dots, Q_r$ vanish simultaneously. Otherwise by virtue of the last hypothesis made in Theorem 1 there would exist a quadratic form $S_i Q_i(z)$ which is positive definite on L , contrary to Lemma B. At this point (z) we have accordingly $m_0 z_k z_k = P(z, S_0) = P(z) > 0$ in view of the first hypothesis of the theorem. It follows that $m_0 > 0$ and hence that $P(z, S_0)$ is positive definite. This proves the first statement in the theorem. The second statement is readily verified. It should be observed that the hypotheses of Theorem 1 imply that there is a point $(z) \neq (0)$ at which the forms $Q_1, \dots, Q_r$ vanish simultaneously. Otherwise the last hypothesis would imply the existence of a positive definite form $S_i Q_i$, contrary to the second hypothesis.

The last sentence in Theorem 1 remains to be established. It is easy to see that if $r=1$ the last hypothesis in §1 is automatically satisfied. Suppose then that $r=1$ and that Q is not indefinite. Let K be the sphere $z_k z_k = 1$. If Q_1 is positive definite, the sum $P(z, S)$ is positive on K , hence is a positive definite form, provided that S_1 is large. If Q_1 is negative definite, $P(z, S)$ is positive definite provided that $-S_1$ is large. If Q_1 is semi-definite, say positive, let L be the set on which Q_1 vanishes. As in Lemma B, this is linear. On LK the form $P(z)$ is positive; it then remains positive on a neighborhood N of LK relative to K . On $K - N$ the form Q_1 is positive. Choose S_1 large enough so that $P(z, S)$ is positive on $K - N$. On N we have $P(z, S) \geq P(z) > 0$, so that $P(z, S)$ is positive on K , and is therefore a positive definite form.

In the proof of the first part of Theorem 1 made above we have established essentially the following more general result:

COROLLARY. *Suppose the last two of the hypotheses made in Theorem 1 hold, and let m be the minimum of the form $P(z)$ on the set of points (z) satisfying the conditions $Q_1(z) = \dots = Q_r(z) = 0$, $z_k z_k = 1$. There exists a set of constants S_i such that the inequality $P(z, S) \geq m z_k z_k$ holds for all points (z) .*

2. A further result. Consider now the case in which the space of points (z) described in the last section is of dimension $2n$. For the purposes of this sec-

tion it will be convenient to denote the i th ($i \leq n$) and the $(n+i)$ th coordinates of (z) by x_i and y_i respectively. Thus the points of our space will be denoted by the symbol (x, y) . To each point (x, y) there is associated a $n \times 2$ -dimensional matrix $(x_i y_i)$ whose i th row is composed of the coordinates x_i, y_i of the point (x, y) . This matrix will be used below to classify the points (x, y) of our space. By a quadratic form in the variables (x, y) will be meant an expression of the form

$$R_{ik}x_ix_k + S_{ik}x_iy_k + T_{ik}y_iy_k.$$

In particular the expression $S_{ik}x_iy_k$ is a quadratic form in the variables (x, y) . Finally by a linear space L of points (x, y) is meant a subspace such that if (x, y) and (x', y') belong to L so also does the point $(ax+bx', ay+by')$, where a and b are arbitrary real constants.

The results described in the last section will be used to prove the following:

THEOREM 2. *Let $P(x, y)$ be a quadratic form in the $2n$ real variables $x_1, \dots, x_n, y_1, \dots, y_n$ with real coefficients. Suppose that the inequality $P(x, y) > 0$ holds whenever the $n \times 2$ -dimensional matrix $(x_i y_i)$ has rank 1. Then there exists an n -rowed skew-symmetric matrix $S = (S_{ik})$ such that the quadratic form $P(x, y) + S_{ik}x_iy_k$ ($i, k = 1, \dots, n$) is positive definite.*

To prove this result let Q_{ik} ($i < k$) be the quadratic form $x_iy_k - x_ky_i$. A linear combination $S_{ik}Q_{ik}$ (summed with $i < k$) is easily seen to be equal to $S_{ik}x_iy_k$ (summed for all i, k) if we set $S_{ii} = 0$, $S_{ki} = -S_{ik}$ ($i < k$). Thus we see that the theorem will be established if we show that the hypotheses of Theorem 1 with Q_i replaced by Q_{ik} are satisfied. The first hypothesis holds since the matrix $(x_i y_i) \neq (0 \ 0)$ has rank 1 if and only if the forms Q_{ik} vanish simultaneously. Moreover a linear combination $S_{ik}Q_{ik}$ with $S \neq 0$ is indefinite. Finally the last hypothesis of Theorem 1 holds by virtue of the following:

THEOREM 3. *Let L be a linear set of points (x, y) such that the $n \times 2$ -dimensional matrix $(x_i y_i)$ has rank 2 at each point $(x, y) \neq (0, 0)$ on L . There exists a skew-symmetric matrix $S = (S_{ik})$ such that the quadratic form $S_{ik}x_iy_k$ is positive definite on L .*

Let m be the dimension of L and let $(X_{1\alpha}, \dots, X_{n\alpha}, Y_{1\alpha}, \dots, Y_{n\alpha})$ ($\alpha = 1, \dots, m$) be a basis for L , that is, a set of m points (X_α, Y_α) in L such that the coordinates of each point (x, y) in L are expressible uniquely in the form

$$(2) \quad x_i = X_{i\alpha}u_\alpha, \quad y_i = Y_{i\alpha}u_\alpha \quad (\alpha = 1, \dots, m).$$

Since the matrix $(x_i y_i)$ has rank 2 at each point $(x, y) \neq (0, 0)$ on L , the matrix

$\lambda X + \mu Y$, where $X = (X_{i\alpha})$ and $Y = (Y_{i\alpha})$, has rank m for every pair of real numbers λ, μ , not both zero. Conversely for every pair of matrices X, Y , having $\lambda X + \mu Y$ of rank m when $(\lambda, \mu) \neq (0, 0)$, the corresponding linear space L defined by equations (2) is such that the matrix $(x_i y_i)$ has rank 2 at each point $(x, y) \neq (0, 0)$ on L . Moreover by the use of equations (2) it is seen that $S_{ik}x_i y_k = R_{\alpha\beta}u_\alpha u_\beta$, where $R = X'SY$ and X' is the transpose of X . Here and elsewhere it is understood that the symbol for an element of a matrix is obtained by adding a pair of subscripts to the symbol for the matrix. Theorem 3 is accordingly equivalent to the following theorem on matrices:

THEOREM 4. *Let X, Y be two $n \times m$ -dimensional matrices such that the matrix $\lambda X + \mu Y$ has rank m for every pair of real numbers λ, μ , not both zero. There exists an n -rowed skew-symmetric matrix S such that the matrix $R = X'SY$ is positive definite, that is, the quadratic form $R_{\alpha\beta}u_\alpha u_\beta$ ($\alpha, \beta = 1, \dots, m$) is positive definite. Here X' is the transpose of X . The matrix R in general will not be symmetric.*

In order to prove Theorem 4 we first observe that in the proof of the theorem we may replace the matrices X, Y by $X_1 = AXB, Y_1 = A Y B$, where A and B are arbitrary nonsingular matrices of dimensions n and m respectively. For, the matrix $\lambda X_1 + \mu Y_1$ has rank m when $(\lambda, \mu) \neq (0, 0)$. Moreover, suppose there exists a skew-symmetric matrix S_1 such that the matrix $X'_1 S_1 Y_1$ is positive definite. Let S be the skew-symmetric matrix $A'S_1 A$, and set $R = X'SY$. Then the matrix $B'RB = X'_1 S_1 Y_1$ is positive definite. It follows readily that R is positive definite and hence that the matrices X, Y can be replaced by X_1, Y_1 respectively. In fact X, Y can be replaced by matrices of the form

$$(3) \quad X = \begin{pmatrix} I \\ O \end{pmatrix}, \quad Y = \begin{pmatrix} C \\ D \end{pmatrix},$$

where I is the m -rowed identity matrix, C is an m -rowed matrix and O is the zero matrix. Suppose that X, Y are of this form. Then the condition that the matrix $\lambda X + \mu Y$ be of rank m for all real numbers λ, μ , not both zero, is equivalent to the condition that the matrix

$$(4) \quad \begin{pmatrix} \lambda I - C \\ -D \end{pmatrix}$$

be of rank m for all real values of λ . Moreover the equation $AXB = X$ holds for two nonsingular matrices A and B if and only if A is of the form

$$(5) \quad A = \begin{pmatrix} B^{-1} & B^{-1}E \\ O & F \end{pmatrix}.$$

When A is of this form we have

$$(6) \quad AYB = \begin{pmatrix} B^{-1}(C + ED)B \\ FDB \end{pmatrix}.$$

In the sequel we shall assume that the matrices X, Y, A are of the forms (3) and (5). It will be understood that if $n = m$ the matrices O, D, E, F do not appear in these matrices. Theorem 4 will be established by replacing Y by a matrix AYB having special properties. This will be done with the help of three lemmas, the first of which is the following:

LEMMA 1. Let $\phi_1(\lambda), \dots, \phi_k(\lambda)$ be the elementary divisors of the matrix $\lambda I - C$ and let ϵ be an arbitrary constant different from zero. With an elementary divisor of the form $\phi_i(\lambda) = [(\lambda - \alpha_i)^2 + \beta_i^2]^{m_i}$ ($\beta_i \neq 0$) associate the $2m_i$ -rowed matrix

$$(7) \quad M_i = \begin{pmatrix} N_i & O & O & \dots & O \\ \epsilon I & N_i & O & \dots & O \\ O & \epsilon I & N_i & \dots & O \\ \cdot & \cdot & \cdot & \dots & \cdot \\ O & O & O & \dots & \epsilon I & N_i \end{pmatrix}, \quad N_i = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

With an elementary divisor of the form $\phi_j(\lambda) = (\lambda - \alpha_j)^{m_j}$ associate the matrix

$$(8) \quad M_j = \begin{pmatrix} \alpha_j & 0 & 0 & \dots & 0 \\ 1 & \alpha_j & 0 & \dots & 0 \\ 0 & 1 & \alpha_j & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 & \alpha_j \end{pmatrix}.$$

There exists a nonsingular m -rowed matrix B such that $B^{-1}CB = C_1$, where

$$(9) \quad C_1 = \begin{pmatrix} M_1 & O & O & \dots & O \\ O & M_2 & O & \dots & O \\ O & O & M_3 & \dots & O \\ \cdot & \cdot & \cdot & \dots & \cdot \\ O & O & O & \dots & M_k \end{pmatrix}.$$

For it is clear from the special forms (7) and (8) that $\phi_i(\lambda) = |\lambda I - M_i|$. If M_i is of the form (8), the $(m_i - 1)$ -rowed minor obtained by deleting the first row and last column has determinant unity. Hence $\phi_i(\lambda)$ is the only non-trivial invariant factor of $\lambda I - M_i$ in this case. Suppose next that M_i is of the

form (7) and let $\phi(\lambda)$ be the determinant of the minor of $\lambda I - M_i$ obtained by deleting the first row and the last column. The values $\phi(\alpha_i + i\beta_i), \phi(\alpha_i - i\beta_i)$ are different from zero since $\epsilon \neq 0$, as one readily verifies. It follows that $\phi(\lambda)$ and $\phi_i(\lambda)$ have no common factors and hence that $\phi_i(\lambda)$ is the only non-trivial invariant factor of $\lambda I - M_i$ when M_i is of the form (7). The matrices $\lambda I - C_1$ and $\lambda I - C$ accordingly have the same elementary divisors and hence the same invariant factors. The matrix C_1 is therefore similar to C , that is, C_1 is of the form $B^{-1}CB$, where B is nonsingular.†

LEMMA 2. *If the equation $|\lambda I - C| = 0$ has no real roots, there exists a matrix C^* similar to C and a skew-symmetric matrix S such that the matrix SC^* is positive definite. The matrix SC^* is not in general symmetric.*

For in this case the diagonal blocks in (9) are of the form (7). Consider C_1 as a function $C_1(\alpha, \beta, \epsilon)$ of the values $\alpha_i, \beta_i, \epsilon$ described in Lemma 1. Let $R(\epsilon) = C_1(0, \beta, \epsilon)$, $S = -R(0)$ and $T = C_1(\alpha, 0, 0)$. Then S and ST are skew-symmetric, S is nonsingular and $C_1 = R(\epsilon) + T$. Since ST is skew-symmetric one has $u_\alpha S_{\alpha\beta} T_{\beta\gamma} u_\gamma = 0$ ($\alpha, \beta, \gamma = 1, \dots, m$) and

$$u_\alpha S_{\alpha\beta} (R_{\beta\gamma} + T_{\beta\gamma}) u_\gamma = u_\alpha S_{\alpha\beta} R_{\beta\gamma} u_\gamma$$

for arbitrary values of (u) . Since $R(0) = -S$ and S is nonsingular, the last quadratic form is positive definite when $\epsilon = 0$ and hence for a value $\epsilon' \neq 0$. The matrices S and $C^* = R(\epsilon') + T$ have the properties described in the lemma.

LEMMA 3. *Let C, D be the matrices appearing in the matrix (4) and suppose the matrix (4) has rank m for all real values of λ . There exists an $m \times (n - m)$ -dimensional matrix E such that the equation*

$$(10) \quad |\lambda I - C - ED| = 0$$

has no real roots if the dimension m of C is even and a single real root if m is odd.

Let us begin by disposing of the case $m = n$. If $m = n$, the conclusion of Lemma 3 is that $|\lambda I - C| \neq 0$ for all real λ . This is an immediate consequence of the hypothesis that matrix (4) has rank $m = n$ for all real λ . Incidentally, if $m = n$, both m and n must be even; otherwise the equation $|\lambda I - C| = 0$ would be of odd degree, and would therefore have a real root.

Suppose then that $m < n$. We shall consider first the case in which $m = 2$ and C, D are of the form

$$(11) \quad C = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} d & e \end{pmatrix}.$$

† A. A. Albert, *Modern Higher Algebra*, University of Chicago Press, 1937, pp. 84-85.

If D is the two-rowed identity matrix, let

$$G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E = G - C.$$

Then the equation (10) takes the form $\lambda^2 + 1 = 0$ and has no real roots. Suppose next that $D = (d \ e)$. We may suppose that $b = 0$ since this can be brought about by replacing λ by $\lambda + b$. Since the matrix (4) has rank 2 for $\lambda = 0$ and $\lambda = a$, we have $e \neq 0$ and $ad + ec \neq 0$. Choose numbers α, β such that

$$\alpha e - \beta a e = -1, \quad \alpha d + \beta e + a = 0.$$

Then equation (10) with $E' = (\alpha \ \beta)$ as the transpose of E reduces to $\lambda^2 + 1 = 0$. The lemma is accordingly true for the matrices (11).

To prove the lemma as stated, let h be the number of real roots of the equation $|\lambda I - C| = 0$, each root counted a number of times equal to its multiplicity. We may suppose that $h \geq 2$. It is sufficient to show that the matrix E can be chosen so that the equation (10) has exactly $h - 2$ real roots. In view of equation (6) it is sufficient to prove this result when C, D are replaced respectively by matrices of the form $C_1 = B^{-1}CB, D_1 = FDB$, where B and F are nonsingular. By virtue of Lemma 1 we may select B so that C_1 is given by (9). In fact B may be chosen so that the matrices (8) corresponding to real roots of $|\lambda I - C| = 0$ have higher subscripts than the matrices (7) corresponding to roots that are not real. It follows that, after a suitable choice of the matrix F in D_1 , one has

$$C_1 = \begin{pmatrix} M & O \\ N & C_2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} U & O \\ V & D_2 \end{pmatrix},$$

where the matrices C_2, D_2 are the matrices C, D in (11). The matrix (4) with $C = C_2, D = D_2$ has rank 2, since otherwise the corresponding matrix with $C = C_1, D = D_1$ could not have rank m . It follows that there exists a matrix E_2 such that the equation (10) with $C = C_2, D = D_2, E = E_2$ has no real roots. Choosing

$$E = \begin{pmatrix} O & O \\ O & E_2 \end{pmatrix},$$

the equation (10) with $C = C_1, D = D_1$ reduces to the product

$$|\lambda I_{m-2} - M| \cdot |\lambda I_2 - C_2 - E_2 D_2| = 0,$$

where the subscript on I denotes its dimension. This equation has $h - 2$ real roots and the proof of Lemma 3 is complete.

We are now in position to prove Theorem 4. As was seen above we can assume that the matrices X, Y are of the form (3). Since we can replace Y

by the matrix (6) we can suppose, by virtue of Lemma 3, that the equation $|\lambda I - C| = 0$ has no real roots if the dimension m of C is even and one real root if m is odd. Consider first the case in which m is even. Then by Lemma 2 there is a nonsingular matrix B and a skew-symmetric matrix S such that the matrix SC^* with $C^* = B^{-1}CB$ is positive definite. Replace Y by AYB , where A is defined by equation (5) with $E=0$ and $F=I$. Then C is replaced by C^* so that the matrix SC is now positive definite. Let S_1 be the n -rowed skew-symmetric matrix

$$\begin{pmatrix} S & O \\ O & O \end{pmatrix}.$$

The product $X'S_1Y$ is then equal to SC and is accordingly positive definite. This proves Theorem 4 for the case in which m is even.

The case in which m is odd will be reduced to the case in which m is even. As was seen above we can assume that the equation $|\beta I - C| = 0$ has only one real root $\lambda = a$. Choose nonsingular matrices B, F such that the matrices $C_1 = B'CB, D_1 = FDB$ are of the form

$$C_1 = \begin{pmatrix} M & O \\ O & a \end{pmatrix}, \quad D_1 = \begin{pmatrix} U & b \\ V & O \end{pmatrix} \quad (b \neq 0).$$

This choice is possible by virtue of Lemma 1 with $M_k = (a)$ and the fact that the matrix (4) has rank m when $\lambda = a$. We can suppose that $C = C_1, D = D_1$, since this result can be brought about by replacing Y by AYB , where A is given by (5) with $E=O$. Clearly $m < n$ since D has at least one row. Let X_1, Y_1 be the matrices

$$X_1 = \begin{bmatrix} I & O & O \\ O & 1 & 0 \\ O & 0 & 1 \\ O & O & O \end{bmatrix}, \quad Y_1 = \begin{bmatrix} M & O & O \\ O & a & -b \\ U & b & a \\ V & O & O \end{bmatrix}$$

obtained by adding a suitable column to each of the matrices X and Y . Since the equations $|\lambda I - M| = 0, (\lambda - a)^2 + b^2 = 0$ have no real roots, the matrix $\lambda X_1 - Y_1$ corresponding to (4) has rank $m+1$ for all real values of λ . As was seen in the last paragraph, there exists, since $m+1$ is even, a skew-symmetric matrix S such that the matrix X'_1SY_1 is positive definite. The matrix $X'SY$, being a principal minor of X'_1SY_1 , is also positive definite. This completes the proof of Theorem 4 and hence also of Theorem 3.

Incidentally, by an argument similar to that just made, it can be shown that if L is a maximal linear space having the properties described in Theorem 3 then its dimension is n if n is even and $n-1$ if n is odd.

3. Application to the calculus of variations. Consider the problem of minimizing the double integral

$$I = \iint_A f(x, y, z_1, \dots, z_n, p_1, \dots, p_n, q_1, \dots, q_n) dx dy,$$

where $p_i = \partial z_i / \partial x$, $q_i = \partial z_i / \partial y$, in a class of subspaces

$$(12) \quad z_i(x, y) \quad ((x, y) \text{ on } A + B; i = 1, \dots, n)$$

having a common boundary, where B is the boundary of A . The integrand $f(x, y, z, p, q)$ is assumed to be continuous and to have continuous first and second partial derivatives on a region $\mathcal{R}$ of points (x, y, z, p, q) . The subspaces (12) are assumed to be continuous, to have continuous second partial derivatives and to have their elements (x, y, z, p, q) in $\mathcal{R}$. The boundary B of A is supposed to be a simply closed continuous curve having a piecewise continuously turning tangent. Weaker differentiability assumptions could be made.

LEMMA 4. Let $S_{ik}(x, y) = -S_{ki}(x, y)$ ($i, k = 1, \dots, n$) be arbitrary continuous functions having continuous first and second derivatives in a neighborhood of the set $A + B$. Then the integral

$$(13) \quad \begin{aligned} J &= \iint_A g(x, y, z, p, q) dx dy \\ &\equiv \iint_A (1/2) \left\{ p_i \frac{\partial}{\partial y} (s_{ik} z_k) - q_i \frac{\partial}{\partial x} (s_{ik} z_k) \right\} dx dy \end{aligned}$$

has the same value for all subspaces (12) here considered.

For by virtue of Green's theorem the value of $2J$ is given by the formula

$$2J = \iint_A \left\{ \frac{\partial}{\partial y} (S_{ik} p_i z_k) - \frac{\partial}{\partial x} (S_{ik} q_i z_k) \right\} dx dy = \int_B z_k S_{ki} dz_i,$$

and hence is completely determined by the common boundary of the subspaces (12).

Let E be a minimizing subspace. Then E must satisfy the *condition of Legendre*,* that is, at each element (x, y, z, p, q) on E the inequality

$$(14) \quad P(\xi, \eta) = f_{p_i p_k} \xi_i \xi_k + 2f_{p_i q_k} \xi_i \eta_k + f_{q_i q_k} \eta_i \eta_k \geq 0$$

must hold for every set (ξ, η) whose $n \times 2$ -dimensional matrix has rank 1. If

* Graves, *The Weierstrass condition for multiple integral variation problems*, Duke Mathematical Journal, vol. 5 (1939), pp. 656-660. Graves proves only the Weierstrass condition. The condition of Legendre is a well known consequence of that of Weierstrass.

the condition (14) with the equality excluded holds on E for every set (ξ, η) whose matrix $(\xi_i \eta_i)$ has rank 1, then E will be said to satisfy the *strengthened condition of Legendre*. It should be emphasized that in this strengthened condition of Legendre we require that the inequality $P(\xi, \eta) > 0$ hold only for sets (ξ, η) whose $n \times 2$ -dimensional matrix $(\xi_i \eta_i)$ has rank 1 and not for all sets $(\xi, \eta) \neq (0, 0)$. We have the following:

THEOREM 5. *Let E be a subspace (12) satisfying the strengthened condition of Legendre. There exists an invariant integral of the form (13) such that for the function $F = f + g$ the inequality*

$$(15) \quad F_{p_i p_k} \xi_i \xi_k + 2F_{p_i q_k} \xi_i \eta_k + F_{q_i q_k} \eta_i \eta_k > 0$$

holds at each element (x, y, z, p, q) on E for every set $(\xi, \eta) \neq (0, 0)$.

For by Theorem 2 there exists for each point on E a skew-symmetric matrix (S_{ik}) such that the quadratic form $P(\xi, \eta) + S_{ik} \xi_i \eta_k$ is positive definite at this point and hence in a neighborhood of this point. It follows readily that the set $A + B$ can be covered by a finite number of circles $C_1, \dots, C_m$ having a common radius r such that to the circle C'_α of radius $2r$ concentric with the circle C_α there is associated a skew-symmetric matrix (S_{ik}^α) of constant elements such that the form $P_\alpha(\xi, \eta) = P(\xi, \eta) + S_{ik}^\alpha \xi_i \eta_k$ is positive definite at each point on E whose projection in the xy -plane is in C'_α . We propose now to construct a set of functions $S_{ik}(x, y) = -S_{ki}(x, y)$ having continuous second derivatives on a neighborhood of $A + B$ and such that the form $P(\xi, \eta) + S_{ik}(x, y) \xi_i \eta_k$ is positive definite on E . To this end let $h(t)$ be a function having continuous second derivatives and such that $h(t) = 0$ when $t \leq r$, $h(t) = 1$ when $t \geq 2r$, and $0 < h(t) < 1$ when $r < t < 2r$. Let (x_α, y_α) be the center of the circle C_α , and let $d_\alpha(x, y)$ be the distance between the points (x, y) and (x_α, y_α) . Set $h_\alpha(x, y) = h[d_\alpha(x, y)]$. It should be observed that the product $h_1 h_2 \dots h_m$ is identically zero on the set $A + B$. This follows because $h_\alpha \equiv 0$ on C_α and the circles $C_1, \dots, C_m$ cover $A + B$. Denote by $S_{ik}(x, y)$ the function

$$(1 - h_1)S_{ik}^1 + h_1(1 - h_2)S_{ik}^2 + \dots + h_1 h_2 \dots h_{m-1}(1 - h_m)S_{ik}^m,$$

where S_{ik}^α ($\alpha = 1, \dots, m$) are the constants described above. The quadratic form $P(\xi, \eta) + S_{ik}(x, y) \xi_i \eta_k$ determined by these functions is easily seen to be identical with the sum

$$(1 - h_1)P_1(\xi, \eta) + h_1(1 - h_2)P_2(\xi, \eta) + \dots + h_1 h_2 \dots h_{m-1}(1 - h_m)P_m(\xi, \eta),$$

where $P_\alpha(\xi, \eta)$ is the quadratic form associated with the circle C' . Each term in this sum is positive or zero, since for each integer α the product

$(1 - h_\alpha)P_\alpha(\xi, \eta)$ is positive definite at points on E corresponding to points in C'_2 and identically zero elsewhere. Moreover the terms do not vanish simultaneously when $(\xi, \eta) \neq (0, 0)$. The quadratic form $P(\xi, \eta) + S_{ik}\xi_i\eta_k$ is accordingly positive definite on E .

We now use these functions S_{ik} to define the function g of equation (13). From the definition of g we at once obtain

$$g(x, y, z, p, q) = (1/2)[(\partial S_{ik}/\partial y)p_i z_k + (\partial S_{ki}/\partial x)q_i z_k] + S_{ik}p_i q_k.$$

This shows that the Legendre quadratic form (15) for g is $S_{ik}\xi_i\eta_k$. Hence if $F = f + g$ the Legendre form (15) for F is exactly the positive definite form $P(\xi, \eta) + S_{ik}\xi_i\eta_k$. This establishes Theorem 5.

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